

More on Matrix Representations of Arbitrary Linear Transformations

On pages 200-201, the author proves Theorem 4.2.2, which appears at the bottom of page 201. The theorem tells us that if $L : V \rightarrow W$ is a linear transformation from an n -dimensional vector space V to an m -dimensional vector space W then, given any choice of bases E and F for V and W , respectively, we can find an $m \times n$ matrix A with the property that $A[\mathbf{v}]_E = [L(\mathbf{v})]_F$ for all $\mathbf{v} \in V$. The argument is at maximum resolution, using a double sum and explicit references to both bases. Here is a simpler (and clearer?) version.

Our basis for V is $E = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n]$. Our basis for W is called F , and we need no details about F . We want $A \in \mathbf{R}^{m \times n}$ such that $A[\mathbf{v}]_E = [L(\mathbf{v})]_F$ for all $\mathbf{v} \in V$. For each $1 \leq j \leq n$, let $\mathbf{a}_j = [L(\mathbf{v}_j)]_F$, and let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$. Let $\mathbf{v} \in V$, and write $\mathbf{v} = \sum_{j=1}^n x_j \mathbf{v}_j$, i.e., $[\mathbf{v}]_E = \mathbf{x}$. It follows that

$$\begin{aligned} L(\mathbf{v}) &= L\left(\sum_{j=1}^n x_j \mathbf{v}_j\right) \text{ (by substitution)} \\ &= \sum_{j=1}^n x_j L(\mathbf{v}_j) \text{ (by linearity),} \end{aligned}$$

but then

$$\begin{aligned} [L(\mathbf{v})]_F &= \left[\sum_{j=1}^n x_j L(\mathbf{v}_j) \right]_F \\ &= \sum_{j=1}^n x_j [L(\mathbf{v}_j)]_F \\ &= A\mathbf{x}. \end{aligned}$$